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A GENERALIZATION TO TURBULENT BOUNDARY LAYERS OF MANGLER's TRANSFORMATION BETWEEN AXISYMMETRIC AND TWO-DIMENSIONAL LAMINAR BOUNDARY LAYERS

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A GENERALIZATION TO TURBULENT BOUNDARY LAYERS OF MANGLER'S TRANSFORMATION BETWEEN AXISYMMETRIC AND TWO-DIMENSIONAL LAMINAR BOUNDARY LAYERS

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ABSTRACT: With the restriction that the friction coefficient is expressible as a power function of the local boundary-layer Reynolds number, the Mangler transformation between axisymmetric and two-dimensional laminar boundary-layer flow is generalized to turbulent flow. Relations are obtained between turbulent boundary-layer quantities in an axisymmetric flow and those at corresponding points in a substitute two-dimensional flow. The transformation is applied to the supersonic turbulent boundary layer on a cone with an attached shock wave and yields simple relations between boundary-layer quantities for a cone and those for a corresponding flat-plate flow. A non-dimensionalization of the equations of continuity, motion, and total-enthalpy gives the variation with Reynolds number of a number of turbulent boundary-layer quantities.

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A Generalization to Turbulent Boundary Layers of Mangler's Transformation Between Axisymmetric and Two-Dimensional Laminar Boundary Layers

This report presents a derivation of a method for obtaining the properties of a turbulent boundary layer on a body of revolution from the properties on a corresponding two-dimensional body. The method is a generalization to turbulent flow of Mangler's well-known transformation for laminar flow; Mangler's transformation is a special case of the generalized transformation.

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LIST OF SYMBOLS

a = tangent of half angle of cone

A_w = surface area of cone

c_f = local friction coefficient $\dfrac{\tau_w}{\frac{\bar{\rho}_e \bar{u}_e^2}{2}}$

c_F = average friction coefficient of cone (eq. 87)

h = enthalpy

H_u = velocity profile shape parameter $\dfrac{\delta u^*}{\theta_u}$

I = total enthalpy, $h + \dfrac{u^2}{2}$

k = constant in friction formula (eq. 42)

L = reference length

M = Mach number

n = exponent in friction formula (eq. 42)

p = static pressure

q = exponent (eq. 45)

Q = $\bar{k} \dfrac{\partial \bar{T}}{\partial y} - \bar{\rho}\ \overline{h'v'}$

r = recovery factor

r_w = radius of body of revolution

s = exponent (eq. 45)

Re_θ = $\dfrac{\bar{u}_e \theta}{\bar{\nu}_e}$

Re_L = $\dfrac{u_\infty L}{\nu_\infty}$

Re_x = $\dfrac{\bar{u}_e x}{\bar{\nu}_e}$

St = local Stanton number $\dfrac{Q_w}{\bar{\rho}_e \bar{u}_e (\bar{I}_a - \bar{I}_w)}$

ST = average Stanton number on cone (eq. 95)

T = temperature

u = velocity parallel to surface

v = velocity perpendicular to surface

x = distance along surface

y = distance from surface

$\beta = \int_o^\delta \frac{\overline{\rho}\,\overline{u}}{\overline{\rho}_e \overline{u}_e} (1 - \frac{\overline{T}}{\overline{T}_e})\, dy$

δ = velocity or temperature boundary-layer thickness, whichever is larger

$\delta^* = \int_o^\delta (1 - \frac{\overline{\rho}\,\overline{u}}{\overline{\rho}_e \overline{u}_e})\, dy$

$\delta_u^* = \int_o^\delta (1 - \frac{\overline{u}}{\overline{u}_e})\, dy$

$\theta = \int_o^\delta \frac{\overline{\rho}\,\overline{u}}{\overline{\rho}_e \overline{u}_e} (1 - \frac{\overline{u}}{\overline{u}_e})\, dy$

$\theta_u = \int_o^\delta \frac{\overline{u}}{\overline{u}_e} (1 - \frac{\overline{u}}{\overline{u}_e})\, dy$

κ = longitudinal curvature of surface

μ = viscosity

$\nu = \frac{\mu}{\rho}$

ρ = density

$\overline{\rho v} = \overline{\rho}\ \overline{v} + \overline{\rho' v'}$

$\tau = \overline{\mu} \frac{\partial \overline{u}}{\partial y} - \overline{\rho}\ \overline{u'v'}$

Ψ = stream function

Superscripts

$\overline{()}$ = time mean value of ( )

$()'$ = difference between time mean and instantaneous value of ( )

Subscripts

$()_a$ = value of ( ) at surface with no heat transfer

$()_e$ = value of ( ) at outer edge of boundary layer

$()_r$ = value of ( ) at reference enthalpy

$()_w$ = value of ( ) at surface

$()_1$ = value of ( ) in two-dimensional flow

$()_\infty$ = value of ( ) in free stream far from body

$()_*$ = non-dimensional value of ( )

INTRODUCTION

In Reference 1, Mangler gives a method for relating the properties of a laminar boundary layer on a body of revolution to those of a corresponding two-dimensional flow. The distribution of the radius along the axis of the body of revolution is used to obtain a distribution of velocity outside the boundary layer in a two-dimensional flow from the distribution of velocity over the body of revolution. The laminar boundary layer is calculated for the velocity distribution in the two-dimensional flow. The properties of the boundary layer on the body of revolution are then obtained from those for the two-dimensional flow.

Up to the present there does not seem to be a corresponding procedure for turbulent flow. The purpose of the present note is to develop one for turbulent flow.

ANALYSIS

Transformation between axisymmetric and two-dimensional flow

The equations of motion, total enthalpy, and continuity, for a turbulent boundary layer over a body of revolution with $\frac{\delta}{r_w} << 1$ and $\kappa\delta << 1$ are (Ref. 2),

$$\bar{\rho}\,\bar{u}\,\frac{\partial\bar{u}}{\partial x} + (\bar{\rho}\,\bar{v} + \overline{\rho' v'})\,\frac{\partial\bar{u}}{\partial y} = -\frac{d\bar{p}}{dx} + \frac{\partial}{\partial y}\left(\bar{\mu}\,\frac{\partial\bar{u}}{\partial y} - \bar{\rho}\,\overline{u'v'}\right) \tag{1}$$

$$\bar{\rho}\,\bar{u}\,\frac{\partial\bar{I}}{\partial x} + (\bar{\rho}\,\bar{v} + \overline{\rho' v'})\,\frac{\partial\bar{I}}{\partial y} = \frac{\partial}{\partial y}\left[\left(\bar{k}\,\frac{\partial\bar{T}}{\partial y} - \bar{\rho}\,\overline{h'v'}\right) + \bar{u}\left(\bar{\mu}\,\frac{\partial\bar{u}}{\partial y} - \bar{\rho}\,\overline{u'v'}\right)\right] \tag{2}$$

$$\frac{\partial r_w \bar{\rho}\,\bar{u}}{\partial x} + \frac{\partial r_w(\bar{\rho}\,\bar{v} + \overline{\rho' v'})}{\partial y} = 0 \tag{3}$$

By analogy with Mangler's transformation, let

$$\frac{\partial \Psi}{\partial y} = \bar{\rho}\,\bar{u}\,r_w \tag{4}$$

$$\frac{\partial \Psi}{\partial x} = -(\bar{\rho}\,\bar{v} + \overline{\rho' v'})r_w \tag{5}$$

$$\frac{\partial \Psi_1}{\partial y_1} = \bar{\rho}_1\,\bar{u}_1 \tag{6}$$

$$\frac{\partial \Psi_1}{\partial x_1} = -(\bar{\rho}_1\,\bar{v}_1 + \overline{\rho_1' v_1'}) \tag{7}$$

$$y_1 L = r_w y \tag{8}$$

$$L\Psi_1(x_1,y_1) = \Psi(x,y) \tag{9}$$

$$\bar{p}_1(x_1) = \bar{p}(x) \tag{10}$$

$$\bar{h}_1(x_1,y_1) = \bar{h}(x,y) \tag{11}$$

$$\bar{I}_1(x_1,y_1) = \bar{I}(x,y) \tag{12}$$

$$\bar{\rho}_1(x_1,y_1) = \bar{\rho}(x,y) \tag{13}$$

$$\bar{\mu}_1(x_1,y_1) = \bar{\mu}(x,y) \tag{14}$$

and generalize Mangler's relation

$$x_1 = \int_0^x \left(\frac{r_w}{L}\right)^2 dx \tag{15}$$

to

$$x_1 = \int_0^x f\left(\frac{r_w}{L}\right) dx \tag{16}$$

then

$$\bar{\rho}\,\bar{u} = \frac{1}{r_w}\frac{\partial\Psi}{\partial y} = \frac{L}{r_w}\frac{\partial\Psi_1}{\partial y} = \frac{L}{r_w}\frac{\partial\Psi_1}{\partial y_1}\frac{\partial y_1}{\partial y}$$

or

$$\bar{\rho}\,\bar{u} = \frac{\partial\Psi_1}{\partial y_1} = \bar{\rho}_1\,\bar{u}_1 \tag{17}$$

Therefore, from (13), it follows that

$$\bar{u}(x,y) = \bar{u}_1(x_1,y_1) \tag{18}$$

Also

$$\frac{\partial(\ )}{\partial x} = \frac{\partial(\ )}{\partial x_1}\frac{\partial x_1}{\partial x} + \frac{\partial(\ )}{\partial y_1}\frac{\partial y_1}{\partial x}$$

or with (8) and (16)

$$\frac{\partial(\ )}{\partial x} = \frac{\partial(\ )}{\partial x_1} f\left(\frac{r_w}{L}\right) + \frac{\partial(\ )}{\partial y_1}\frac{y_1}{r_w}\frac{dr_w}{dx} \tag{19}$$

Also

$$\bar{\rho}\,\bar{v} + \overline{\rho'v'} = -\frac{1}{r_w}\frac{\partial\Psi}{\partial x} = -\frac{L}{r_w}\frac{\partial\Psi_1}{\partial x}$$

or, with (6), (7), and (19)

$$\bar{\rho}\,\bar{v} + \overline{\rho'v'} = \frac{L}{r_w}\left(\bar{\rho}_1\,\bar{v}_1 + \overline{\rho_1'v_1'}\right) f\left(\frac{r_w}{L}\right) - \bar{\rho}_1\,\bar{u}_1\frac{Ly_1}{r_w^2}\frac{dr_w}{dx} \tag{20}$$

Also from (8),

$$\frac{\partial(\ )}{\partial y} = \frac{r_w}{L}\frac{\partial(\ )}{\partial y_1} \tag{21}$$

Then

$$\bar{\rho}\ \bar{u}\ \frac{\partial(\ )}{\partial x} + (\bar{\rho}\ \bar{v} + \overline{\rho' v'})\ \frac{\partial(\ )}{\partial y} = f(\frac{r_w}{L})\,[\bar{\rho}_1\ \bar{u}_1\frac{\partial(\ )}{\partial x_1} + (\bar{\rho}_1\bar{v}_1 + \overline{\rho_1' v_1'})\frac{\partial(\ )}{\partial y_1}] \tag{22}$$

Also, from (16),

$$\frac{d\bar{p}}{dx} = \frac{d\bar{p}_1}{dx_1}\frac{dx_1}{dx} = \frac{d\bar{p}_1}{dx_1}\ f(\frac{r_w}{L}) \tag{23}$$

Far enough from the surface at sufficiently large Reynolds numbers the term $\rho\ \overline{u'v'}$ is much larger than $\bar{\mu}\ \frac{\partial\bar{u}}{\partial y}$ so that the term $(\bar{\mu}\ \frac{\partial\bar{u}}{\partial y} - \bar{\rho}\ \overline{u'v'})$, which is the total shear stress τ, is practically equal to the turbulent stress $-\bar{\rho}\ \overline{u'v'}$. At the wall, the shear is $\bar{\mu}\ \frac{\partial\bar{u}}{\partial y}$. The magnitude of the surface stress $\bar{\mu}\ \frac{\partial\bar{u}}{\partial y}$ is, however, fixed by the turbulent shear stress $-\bar{\rho}\ \overline{u'v'}$ further out. Consequently, the total shear stress τ is assumed to behave like a turbulent shear everywhere in the boundary layer. Then with (18), (21), (22) and (23), (1) becomes

$$f(\frac{r_w}{L})\,[\bar{\rho}_1\bar{u}_1\ \frac{\partial\bar{u}_1}{\partial x_1} + (\bar{\rho}_1\bar{v}_1 + \overline{\rho_1' v_1'}\ \frac{\partial\bar{u}_1}{\partial y_1}] = -\ \frac{d\bar{p}_1}{dx_1}\ f(\frac{r_w}{L}) + (\frac{r_w}{L})\ \frac{\partial\tau}{\partial y_1} \tag{24}$$

To remove the $(\frac{r_w}{L})$ terms it is sufficient that

$$\tau = \tau_1\ \frac{f(\frac{r_w}{L})}{(\frac{r_w}{L})} \tag{25}$$

Equation (24), then becomes

$$\bar{\rho}_1 \bar{u}_1 \frac{\partial \bar{u}_1}{\partial x_1} + (\bar{\rho}_1 \bar{v}_1 + \overline{\rho_1' v_1'}) \frac{\partial \bar{u}_1}{\partial y} = - \frac{d\bar{p}_1}{dx_1} + \frac{\partial \tau_1}{\partial y_1} , \tag{26}$$

the equation for two-dimensional flow.

Equation (2) for the total enthalpy is transformed in the same way. The argument for the assumption that the total shear stress τ behaves like a turbulent shear everywhere in the boundary layer is also applied to the total energy transfer Q with the result that Q is assumed to behave like a turbulent energy transfer everywhere in the boundary layer. Then (2) becomes

$$\bar{\rho}_1 \bar{u}_1 \frac{\partial \bar{I}_1}{\partial x_1} + (\bar{\rho}_1 \bar{v}_1 + \overline{\rho_1' v_1'}) \frac{\partial \bar{I}_1}{\partial y_1} = \frac{\partial}{\partial y_1} (Q_1 + u_1 \tau_1) \tag{27}$$

where

$$Q = Q_1 \frac{f(\frac{r_w}{L})}{(\frac{r_w}{L})} \tag{28}$$

By use of (16), (19), (20), and (21), the equation of continuity (3) becomes

$$\frac{\partial r_w \bar{\rho}_1 \bar{u}_1}{\partial x_1} f(\frac{r_w}{L}) + (\frac{\partial r_w \bar{\rho}_1 \bar{u}_1}{\partial y_1}) \frac{y_1}{r_w} \frac{dr_w}{dx}$$

$$+ \frac{r_w}{L} \frac{\partial}{\partial y_1} [r_w \frac{L}{r_w} f(\frac{r_w}{L}) (\bar{\rho}_1 \bar{v}_1 + \overline{\rho_1' v_1'}) - \bar{\rho}_1 \bar{u}_1 \frac{L y_1}{r_w} \frac{dr_w}{dx}]$$

$$= 0$$

or

$$\frac{\partial \bar{\rho}_1 \bar{u}_1}{\partial x_1} + \frac{\partial (\bar{\rho}_1 \bar{v}_1 + \overline{\rho_1' v_1'})}{\partial y_1} = 0 , \tag{29}$$

the continuity equation for two-dimensional flow.

If the boundary-layer equation of motion (1) is integrated with respect to y from y = 0 to y = δ and the continuity equation (3) is used, the result is the momentum equation for the boundary layer on a body of revolution, namely,

$$\frac{d\theta}{dx} + \frac{\theta}{\bar{u}_e}\frac{d\bar{u}_e}{dx}\left(2 + \frac{\delta^*}{\theta}\right) + \frac{\theta}{\bar{\rho}_e}\frac{d\bar{\rho}_e}{dx} + \frac{\theta}{r_w}\frac{dr_w}{dx} - \frac{(\overline{\rho v})_w}{\bar{\rho}_e\bar{u}_e} = \frac{\tau_w}{\bar{\rho}_e\bar{u}_e^{\,2}} \tag{30}$$

Equation (30) can be written as

$$\frac{d\theta r_w}{dx} + \frac{\theta r_w}{\bar{u}_e}\frac{d\bar{u}_e}{dx}\left(2 + \frac{\delta^*}{\theta}\right) + \frac{\theta r_w}{\bar{\rho}_e}\frac{d\bar{\rho}_e}{dx} - \frac{(\overline{\rho v})_w r_w}{\bar{\rho}_e\bar{u}_e} = \frac{\tau_w}{\bar{\rho}_e\bar{u}_e^{\,2}}\, r_w \tag{31}$$

From (8), (13) and (18) and the definition of θ and θ_1, namely

$$\theta = \int_0^{\delta} \frac{\bar{\rho}\,\bar{u}}{\bar{\rho}_e\bar{u}_e}\left(1 - \frac{\bar{u}}{\bar{u}_e}\right) dy$$

and

$$\theta_1 = \int_0^{\delta_1} \frac{\bar{\rho}_1\bar{u}_1}{\bar{\rho}_{e_1}\bar{u}_{e_1}}\left(1 - \frac{\bar{u}_1}{\bar{u}_{e_1}}\right) dy_1$$

it follows that

$$\theta r_w = \theta_1 L \tag{32}$$

In the same way it follows that

$$\delta^* r_w = \delta_1^* L \tag{33}$$

and therefore that

$$\frac{\delta^*}{\theta} = \frac{\delta_1^*}{\theta_1} \tag{34}$$

By using (13), (16), (25), (32), (33), (34), and (20) with y = 0, equation (31) becomes the two-dimensional momentum equation,

$$\frac{d\theta_1}{dx_1} + \frac{\theta_1}{\bar{u}_{e_1}} \frac{d\bar{u}_{e_1}}{dx_1} \left(2 + \frac{\delta_1^*}{\theta_1}\right) + \frac{\theta_1}{\bar{\rho}_{e_1}} \frac{d\bar{\rho}_{e_1}}{dx_1} - \frac{(\overline{\rho v})_{1w}}{\bar{\rho}_{e_1}\bar{u}_{e_1}} = \frac{\tau_{w_1}}{\bar{\rho}_{e_1}\bar{u}_{e_1}^{\ 2}} \tag{35}$$

By integrating the boundary-layer total-enthalpy equation, (2), from y = 0 to y = δ and using the equation of continuity (3), there is obtained the integral total-enthalpy equation for a body of revolution, namely,

$$\frac{d\beta}{dx} + \frac{\beta}{\bar{I}_e}\frac{d\bar{I}_e}{dx} + \frac{\beta}{\bar{u}_e}\frac{d\bar{u}_e}{dx} + \frac{\beta}{\bar{\rho}_e}\frac{d\bar{\rho}_e}{dx} + \frac{\beta}{r_w}\frac{dr_w}{dx} + \frac{(\delta^*-\delta)}{\bar{I}_e}\frac{d\bar{I}_e}{dx}$$

$$+ \left(\frac{\bar{I}_w}{\bar{I}_e} - 1\right) \frac{(\overline{\rho v})_w}{\bar{\rho}_e\bar{u}_e} = \frac{1}{\bar{\rho}_e\bar{u}_e\bar{I}_e}\, Q_w \tag{36}$$

From (8), (12), (13), (18), and the definition of β and β_1, namely

$$\beta = \int_0^{\delta} \frac{\bar{\rho}\,\bar{u}}{\bar{\rho}_e\bar{u}_e}\left(1 - \frac{\bar{I}}{\bar{I}_e}\right) dy$$

and

$$\beta_1 = \int_0^{\delta_1} \frac{\bar{\rho}_1\bar{u}_1}{\bar{\rho}_{e_1}\bar{u}_{e_1}}\left(1 - \frac{\bar{I}_1}{\bar{I}_{e_1}}\right) dy_1$$

it follows that

$$\beta r_w = \beta_1 L \tag{37}$$

Then, with the same procedure that was used to obtain (35) from (31), it follows that

$$\frac{d\beta_1}{dx_1} + \frac{\beta_1}{\bar{I}_{e_1}} \frac{d\bar{I}_{e_1}}{dx_1} + \frac{\bar{\beta}_1}{\bar{u}_{e_1}} \frac{d\overline{u}_{e_1}}{dx_1} + \frac{\beta_1}{\overline{\rho}_{e_1}} \frac{d\overline{\rho}_{e_1}}{dx_1} + \frac{(\delta_1{}^*-\delta_1)}{\bar{I}_{e_1}} \frac{d\bar{I}_{e_1}}{dx_1}$$

$$+ \left(\frac{\bar{I}_{w_1}}{\bar{I}_{e1}} - 1\right) \frac{(\overline{\rho v})1_w}{\overline{\rho}_{e_1}\overline{u}_{e_1}} = \frac{1}{\overline{\rho}_{e_1}\overline{u}_{e_1}\bar{I}_{e_1}} Q1_w \quad , \tag{38}$$

the integral total-enthalpy equation for two-dimensional flow. The integral kinetic energy and other integral equations for flow over a body of revolution can be transformed to two-dimensional equations in the same way. The two-dimensional equations can, of course, also be obtained directly from (26), (27) and (29).

Up to this point the function $f(\frac{r_w}{L})$ in (16) has not been specified. A method of determining $f(\frac{r_w}{L})$ is to write (25) for $y = 0$, thus

$$\tau_w = \tau 1_w \frac{f(\frac{r_w}{L})}{(\frac{r_w}{L})} \tag{39}$$

By using (13) and (18), (39) can be written as

$$f(\frac{r_w}{L}) = \frac{r_w}{L} \frac{C_f}{C_{f1}} \tag{40}$$

The friction coefficient is now expressed as a function of Re_θ, M_e, $\frac{T_w}{T_e}$, and the shape of the non-dimensional velocity profile. The function is taken to be the same for axisymmetric as for two-dimensional flow. That is, $\frac{\delta}{r_w}$ is assumed to be so small that any effect of transverse curvature on the boundary-layer flow that is independent of Re_θ, M_e, $\frac{T_w}{T_e}$, and the non-dimensional velocity profile is negligible. Then

$$C_f = F(Re_\theta, M_e, \frac{T_w}{T_e}, H_u)$$

and

$$C_{f_1} = F(Re_{\theta_1}, M_{e_1}, \frac{T_{w_1}}{T_{e_1}}, H_{u_1})$$

where $H_u (= \frac{\delta^*_u}{\theta_u})$ denotes the effects of the shape of the velocity profile; its use does not imply that the profiles are actually a single-parameter family with H_u as the parameter. From (40) it follows that

$$f(\frac{r_w}{L}) = \frac{r_w}{L} \frac{F(Re_\theta, M_e, \frac{T_w}{T_e}, H_u)}{F(Re_{\theta_1}, M_{e_1}, \frac{T_{w_1}}{T_{e_1}}, H_{u_1})} \tag{41}$$

where $Re_\theta = Re_{\theta_1} \frac{L}{r_w}$, from (32).

If a power formula approximation is taken for the friction coefficient then

$$\frac{C_f}{2} = \frac{k}{Re_\theta^{\ n}} \tag{42}$$

and

$$\frac{C_{f_1}}{2} = \frac{k}{Re_{\theta_1}^{\ n}} \tag{43}$$

where $k = k\ (\frac{\delta_u^*}{\theta_u}, M_e, \frac{T_w}{T_e})$. But at corresponding x and x_1, $\frac{\delta_u^*}{\theta_u} = \frac{\delta_{u_1}^*}{\theta_{u_1}}$, $M_e = M_{e_1}$, $T_e = T_{e_1}$, and $T_w = T_{w_1}$. Therefore, k in (42) and (43) are equal at corresponding x and x_1. Then from (40)

$$f\left(\frac{r_w}{L}\right) = \frac{r_w}{L}\left(\frac{\theta_1}{\theta}\right)^n$$

or with (32)

$$f\left(\frac{r_w}{L}\right) = \left(\frac{r_w}{L}\right)^{n+1} \tag{44}$$

For laminar flow, n = 1 and Mangler's transformation, (15), is obtained.

Non-dimensional form of equations

Equations (1), (2), and (3) can be written in a non-dimensional form so that the Reynolds number does not appear, just as for laminar flow (Ref. 3). Thus let

$$\begin{aligned}
x_* &= \frac{x}{L} & & \\
r_{w_*} &= \frac{r_w}{L} & & \\
y_* &= \frac{y}{L}\,Re_L{}^q & \tau_* &= \frac{\tau}{\rho_\infty u_\infty{}^2}\,Re_L{}^s \\
u_* &= \frac{\bar{u}}{u_\infty} & I_* &= \frac{\bar{I}}{I_\infty} \\
v_* &= \frac{\bar{v}}{u_\infty}\,Re_L{}^q & Q_* &= \frac{Q}{\rho_\infty u_\infty I_\infty}\,Re_L{}^s \\
(\overline{\rho' v'})_* &= \frac{\overline{\rho' v'}}{\rho_\infty u_\infty}\,Re_L{}^q & & \\
\rho_* &= \frac{\bar{\rho}}{\rho_\infty} & & \\
\mu_* &= \frac{\bar{\mu}}{\mu_\infty} & & \\
p_* &= \frac{\bar{p}-p_\infty}{\rho_\infty u_\infty{}^2} & &
\end{aligned} \tag{45}$$

Then (1) becomes, with $(\bar{\mu}\frac{\partial\bar{u}}{\partial y} - \bar{\rho}\ \overline{u'v'}) = \tau$,

$$\rho_*\rho_\infty u_* u_\infty \frac{\partial u_* u_\infty}{\partial x_* L} + [\rho_*\rho_\infty v_* u_\infty Re_L{}^{-q} + (\overline{\rho' v'})_*\rho_\infty u_\infty Re_L{}^{-q}] \frac{\partial u_* u_\infty}{\partial y_* L Re_L{}^{-q}}$$

$$= - \frac{dp_*}{dx_* L}\rho_\infty u_\infty{}^2 + \frac{\partial\tau_*}{\partial y_* L Re_L{}^{-q}} \rho_\infty u_\infty{}^2 Re_L{}^{-s} \tag{46}$$

Equation (46) is independent of Re_L if

$$s = q \tag{47}$$

Then (46) becomes

$$\rho_* u_* \frac{\partial u_*}{\partial x_*} + [\rho_* v_* + (\overline{\rho' v'})_*] \frac{\partial u_*}{\partial y_*} = - \frac{dp_*}{dx_*} + \frac{\partial\tau_*}{\partial y_*}, \tag{48}$$

an equation independent of Re_L. Now let

$$\frac{\tau_w}{\rho_e u_e{}^2} = \frac{k}{Re_\theta{}^n}$$

or by use of (45),

$$\tau_{*w} = \frac{\frac{\rho_e}{\rho_\infty}(\frac{u_e}{u_\infty})^2 k\ Re_L{}^q}{(\frac{u_{*e}\rho_{*e}\theta_*}{\mu_{*e}})^n (\frac{u_\infty\rho_\infty L}{\mu_\infty})^n Re_L{}^{-nq}}$$

For τ_{*w} to be independent of Re_L it is necessary that

$$q = \frac{n}{1+n} \tag{49}$$

Equation (27), with $Q = \bar{k}\frac{\partial\bar{T}}{\partial y} - \bar{\rho}\ \overline{v'h'}$ and (45), becomes

$$\rho_* u_* \frac{\partial I_*}{\partial x_*} + [\rho_* v_* + (\overline{\rho' v'})_*] \frac{\partial I_*}{\partial y_*} = \frac{\partial Q_*}{\partial y_*} + \frac{u_\infty{}^2}{I_\infty}\frac{\partial(u_*\tau_*)}{\partial y_*} \tag{50}$$

The equation of continuity, (3), becomes

$$\frac{\partial r_{w*}\rho_*}{\partial x_*} + \frac{\partial r_{w*}[\rho_* v_* + (\overline{\rho' v'})_*]}{\partial y_*} = 0 \tag{51}$$

For two-dimensional flow, $r_{w_*} = 1$ in (51).

The conditions at the outer edge of the boundary layer become

$$u_{*e}(x_*),\ I_{*e}(x_*),\ p_{*e}(x_*) \tag{52}$$

and at the surface

$$u_* = 0,\ v_* = v_*(x_*),\ I_{*w} = I_{*w}(x_*) \text{ or } Q_{*w} = Q_{*w}(x_*) \tag{53}$$

Because (48), (50), and (51) are independent of Reynolds number it follows from (45) and (49) that both the velocity profile $\frac{\bar{u}}{u_\infty}$ or $\frac{\bar{u}}{u_e}$ and the total-enthalpy profile $\frac{\bar{I}}{I_\infty}$ or $\frac{\bar{I}}{I_e}$ are fixed functions of $\frac{y}{L}\,Re_L^{\frac{n}{n+1}}$ at a fixed $\frac{x}{L}$ when the boundary conditions given by (52) and (53) are independent of Reynolds number and $\frac{u_\infty^2}{I_\infty}$ is fixed. The indication is also that $\frac{\theta}{L}\,Re_L^{\frac{n}{n+1}}$, $\frac{\delta^*}{L}\,Re_L^{\frac{n}{n+1}}$, and $\frac{\beta}{L}\,Re_L^{\frac{n}{n+1}}$ are independent of Re_L. Moreover, the local shear stress and local heat transfer vary as $Re_L^{-\frac{n}{n+1}}$; therefore, the total friction stress and the total heat transfer vary as $Re_L^{-\frac{n}{n+1}}$ if the boundary layer is either entirely laminar or entirely turbulent. Because the profile drag of a body is proportional to $\frac{\theta}{L}$ at the trailing edge, the profile drag also varies as $Re_L^{-\frac{n}{n+1}}$ if the boundary layer is entirely laminar or entirely turbulent.

Because τ_* is independent of Reynolds number, the indication is that the skin friction drops to zero at a point that is independent of Reynolds number. Therefore, the separation point is also independent of Reynolds number. The same conclusion follows from the conclusion that the non-dimensional velocity profiles are independent of Reynolds number. A requirement is that the friction coefficient must be expressible as a power function of Re_θ. Moreover, the non-dimensional velocity profiles and the non-dimensional thicknesses at the initial point of the boundary layer must be independent of Reynolds number.

If in (53) $Q_{*w} = 0$ is used as a boundary condition, $\frac{u_\infty^2}{I_\infty}$ fixed, and the other boundary conditions made independent of Re_L, a solution of (48), (50), and (51) gives $I_*(x_*,y_*)$ and thus $I_*(x_*,0)$ or $I_{*a}(x_*)$. Under these conditions $I_{*a}(x_*)$ is independent of Re_L. From the definition of the recovery factor r,

$$h_a = h_e + r\,\frac{u_e^2}{2} \tag{54}$$

it follows that

$$I_{*a}(x_*) = I_{*e}(x_*) - \frac{u_{*e}^2}{2}\,(1-r)\,\frac{u_\infty^2}{I_\infty} \quad . \tag{55}$$

Therefore, when $\frac{u_\infty^2}{I_\infty}$ is fixed and the boundary conditions are independent of Re_L, it follows that $r(x_*)$ is independent of Re_L.

Relation between boundary-layer quantities in axisymmetric and in two-dimensional flow

From (45), (47), and (49) it follows that, with "bars" dropped,

$$\frac{\tau_w}{\rho_e u_e^2} Re_L^{\frac{n}{n+1}} = \frac{\tau_{*w}(\frac{x}{L})}{(\frac{\rho_e}{\rho_\infty})(\frac{u_e}{u_\infty})^2} \tag{56}$$

or

$$\frac{\tau_w}{\rho_e u_e^2} Re_x^{\frac{n}{n+1}} = \frac{(\frac{x}{L})^{\frac{n}{n+1}} \tau_{*w}(\frac{x}{L})}{(\frac{\rho_e}{\rho_\infty})(\frac{u_e}{u_\infty})^2} = g(x/L) \tag{57}$$

The R.H.S. of (56) and (57) are independent of Reynolds number Re_L if $\frac{u_\infty^2}{I_\infty}$ and the boundary conditions given by (52) and (53) are independent of Re_L. Therefore, under these conditions the L.H.S. are also independent of the Reynolds number Re_L.

The quantity $\frac{\tau_w}{\rho_e u_e^2} Re_x^{\frac{n}{n+1}}$ can be written as

$$\frac{\tau_w}{\rho_e u_e^2} Re_x^{\frac{n}{n+1}} = \frac{\tau_{w_1}(\frac{r_w}{L})^n}{\rho_{e_1} u_{e_1}^2} \left(\frac{u_{e_1} x_1}{\nu_{e_1}}\right)^{\frac{n}{n+1}} \left(\frac{x}{x_1}\right)^{\frac{n}{n+1}}$$

or

$$C_{f_1} Re_{x_1}^{\frac{n}{n+1}} = C_f Re_x^{\frac{n}{n+1}} \left[\frac{\int_o^{\frac{x}{L}} (\frac{r_w}{L})^{n+1} d\,\frac{x}{L}}{\frac{x}{L}(\frac{r_w}{L})^{n+1}}\right]^{\frac{n}{n+1}} \tag{58}$$

at corresponding x and x_1. Recall that at corresponding x and x_1,

$M_e = M_{e_1}$, $T_e = T_{e_1}$, and $T_w = T_{w_1}$. Also, from (32) and (33)

$$\frac{\theta_1}{x_1} Re_{x_1}^{\frac{n}{n+1}} = \frac{\theta}{x} Re_x^{\frac{n}{n+1}} \left[\frac{\frac{x}{L}(\frac{r_w}{L})^{n+1}}{\int_0^{\frac{x}{L}} (\frac{r_w}{L})^{n+1} d\, \frac{x}{L}} \right]^{\frac{1}{n+1}} \tag{59}$$

and

$$\frac{\delta_1^*}{x_1} Re_{x_1}^{\frac{n}{n+1}} = \frac{\delta^*}{x} Re_x^{\frac{n}{n+1}} \left[\frac{\frac{x}{L}(\frac{r_w}{L})^{n+1}}{\int_0^{\frac{x}{L}} (\frac{r_w}{L})^{n+1} d\, \frac{x}{L}} \right]^{\frac{1}{n+1}} \tag{60}$$

at corresponding x and x_1.

From (25) and (28) it follows that

$$\frac{Q}{\tau} = \frac{Q_1}{\tau_1}$$

or

$$\frac{Q_w}{\tau_w} = \frac{Q_{1_w}}{\tau_{1_w}} \tag{61}$$

The Reynolds analogy factor is thus the same for corresponding x and x_1 locations in the axisymmetric and in the substitute two-dimensional flow.

The Stanton number St is, with "bars" dropped,

$$St = \frac{Q_w}{\rho_e u_e (I_a - I_w)} = \frac{Q_{*w} \rho_\infty u_\infty I_\infty Re_L^{-s}}{\rho_e u_e (I_a - I_w)} \tag{62}$$

or

$$St\, Re_L^{\frac{n}{n+1}} = Q_{*w} \left(\frac{\rho_\infty}{\rho_e}\right) \left(\frac{u_\infty}{u_e}\right) \frac{1}{I_{*a} - I_{*w}} \tag{63}$$

or

$$St\ Re_x^{\frac{n}{n+1}} = Q_{*w}\left(\frac{\rho_\infty}{\rho_e}\right)\left(\frac{u_\infty}{u_e}\right)\frac{1}{I_{*a}-I_{*w}}\left(\frac{x}{L}\right)^{\frac{n}{n+1}} = G\left(\frac{x}{L}\right) \tag{64}$$

The R.H.S. of (63) and (64) are independent of Re_L if $\frac{u_\infty^2}{I_\infty}$, and the boundary conditions given by (52) and (53) are independent of Re_L. Therefore, with these restrictions the L.H.S. are also independent of Re_L.

From the definition of St and with (12), (13), (18), (28), and (44), the quantity $St\ Re_x^{\frac{n}{n+1}}$ can be written as

$$St\ Re_x^{\frac{n}{n+1}} = St_1\ Re_{x_1}^{\frac{n}{n+1}}\left(\frac{r_w}{L}\right)^n\left(\frac{x}{x_1}\right)^{\frac{n}{n+1}}$$

or

$$St_1\ Re_{x_1}^{\frac{n}{n+1}} = St\ Re_x^{\frac{n}{n+1}}\left[\frac{\int_o^{\frac{x}{L}}\left(\frac{r_w}{L}\right)^{n+1} d\,\frac{x}{L}}{\frac{x}{L}\left(\frac{r_w}{L}\right)^{n+1}}\right]^{\frac{n}{n+1}} \tag{65}$$

By use of (37), there follows

$$\frac{\beta_1}{x_1}\ Re_{x_1}^{\frac{n}{n+1}} = \frac{\beta\left(\frac{r_w}{L}\right)}{x_1}\left(\frac{u_{e_1}x_1}{\nu_{e_1}}\right)^{\frac{n}{n+1}}$$

or

$$\frac{\beta_1}{x_1}\ Re_{x_1}^{\frac{n}{n+1}} = \frac{\beta}{x}\ Re_x^{\frac{n}{n+1}}\left[\frac{\frac{x}{L}\left(\frac{r_w}{L}\right)^{n+1}}{\int_o^{\frac{x}{L}}\left(\frac{r_w}{L}\right)^{n+1} d\,\frac{x}{L}}\right]^{\frac{1}{n+1}} \tag{66}$$

Application to cones in supersonic flow with attached nose shock

For a cone $\frac{r_w}{L} = a\frac{x}{L}$ and ρ_e and u_e are independent of x. Then it follows from (13) and (18) that $\rho_{e_1} = \rho_e$ and $u_{e_1} = u_e$; therefore, ρ_{e_1} and u_{e_1} are also independent of x. Thus the corresponding two-dimensional flow is that over a flat plate at zero angle of attack. For this case equation (58) becomes

$$C_{f_1} \ Re_{x_1}^{\frac{n}{n+1}} = C_f \ Re_x^{\frac{n}{n+1}} \ (n+2)^{-\frac{n}{n+1}} \tag{67}$$

at corresponding x and x_1, with $M_e = M_{e_1}$, $T_e = T_{e_1}$, $\nu_e = \nu_{e_1}$, $T_w = T_{w_1}$. Also, from (59)

$$\frac{\theta_1}{x_1} \ Re_{x_1}^{\frac{n}{n+1}} = \frac{\theta}{x} \ Re_x^{\frac{n}{n+1}} \ (n+2)^{\frac{1}{n+1}} \tag{68}$$

and from (60)

$$\frac{\delta_1^*}{x_1} \ Re_{x_1}^{\frac{n}{n+1}} = \frac{\delta^*}{x} \ Re_x^{\frac{n}{n+1}} \ (n+2)^{\frac{1}{n+1}} \tag{69}$$

For a plate or cone with constant surface temperature the reference length L can be replaced by x. Therefore, the conclusion from (48), (50), (51), (52), and (53) that the local shear stress varies as $Re_L^{-\frac{n}{n+1}}$ can be expressed as

$$C_{f_1} \ Re_{x_1}^{\frac{n}{n+1}} = C_1 \tag{70}$$

and

$$C_f \ Re_x^{\frac{n}{n+1}} = C_2 \tag{71}$$

Consequently, $C_{f_1} \, Re_{x_1}^{\frac{n}{n+1}}$ is independent of x_1 and $C_f \, Re_x^{\frac{n}{n+1}}$ is independent of x. Therefore, the relation obtained from (67), namely,

$$\left(\frac{C_f}{C_{f_1}}\right) \left(\frac{Re_x}{Re_{x_1}}\right)^{\frac{n}{n+1}} = (n+2)^{\frac{n}{n+1}} \qquad (72)$$

holds for all combinations of x and x_1. In particular for $Re_x = Re_{x_1}$, (72) becomes

$$\frac{C_f}{C_{f_1}} = (n+2)^{\frac{n}{n+1}} \qquad (73)$$

and for $C_f = C_{f_1}$, (72) becomes

$$\frac{Re_{x_1}}{Re_x} = \frac{1}{n+2} \qquad (74)$$

For laminar flow n = 1 and (73) becomes the well-known relation

$$\frac{C_f}{C_{f_1}} = \sqrt{3} \qquad (75)$$

and (74) becomes

$$\frac{Re_{x_1}}{Re_x} = \frac{1}{3} \qquad (76)$$

In Reference (4) Van Driest obtains the relation

$$\frac{Re_{x_1}}{Re_x} = \frac{1}{2} \qquad (77)$$

for turbulent flow instead of (74). In going from equation (16) to (17) and from (19) to (20) of Reference (4) it was assumed that a quantity called "a" was very large. This means that C_f was assumed to approach zero. From (42) or (43), this occurs when $Re_\theta \to \infty$. From the character of friction formulas (see Ref. 5 for example), $n \to 0$ as $Re_\theta \to \infty$. Then (74) approaches (77). Consequently, (77) is a limiting relation for $Re_\theta \to \infty$.

For a plate or cone with constant surface temperature, the conclusion from (48), (50), (51), (52), and (53) that

$\frac{\theta}{L} Re_L^{\frac{n}{n+1}}$ is independent of Re_L becomes

$$\frac{\theta_1}{x_1} Re_{x_1}^{\frac{n}{n+1}} = const_1$$

and

$$\frac{\theta}{x} Re_x^{\frac{n}{n+1}} = const_2$$

Therefore, $\frac{\theta_1}{x_1} Re_{x_1}^{\frac{n}{n+1}}$ is independent of x_1 and $\frac{\theta}{x} Re_x^{\frac{n}{n+1}}$ is independent of x. Consequently, the relation obtained from (68), namely,

$$\frac{(\frac{\theta_1}{x_1})}{(\frac{\theta}{x})} \left(\frac{Re_{x_1}}{Re_x}\right)^{\frac{n}{n+1}} = (n+2)^{\frac{1}{n+1}} \tag{78}$$

holds for all combinations of x and x_1. In particular, for $x = x_1$, and $Re_x = Re_{x_1}$ (78) becomes

$$\frac{\theta_1}{\theta} = (n+2)^{\frac{1}{n+1}} \tag{79}$$

In the same way

$$\frac{\delta_1^*}{\delta^*} = (n+2)^{\frac{1}{n+1}} \qquad (80)$$

From (65) it follows that when the body of revolution is a cone,

$$St_1 \, Re_{x_1}^{\frac{n}{n+1}} = St \, Re_x^{\frac{n}{n+1}} (n+2)^{-\frac{n}{n+1}} \qquad (81)$$

and from (66) that,

$$\frac{\beta_1}{x_1} Re_{x_1}^{\frac{n}{n+1}} = \frac{\beta}{x} Re_x^{\frac{n}{n+1}} (n+2)^{\frac{1}{n+1}} \qquad (82)$$

From (67) and (81) it follows that

$$\frac{St_1}{C_{f_1}} = \frac{St}{C_f} \qquad (83)$$

or

$$\frac{St}{St_1} = \frac{C_f}{C_{f_1}} \qquad (84)$$

The relation between C_f and C_{f_1} for $Re_x = Re_{x_1}$ is given by (73); therefore, for $Re_x = Re_{x_1}$ (84) becomes

$$\frac{St}{St_1} = (n+2)^{\frac{n}{n+1}} \qquad (85)$$

By dividing (82) by (81), then using (85), which requires $Re_x = Re_{x_1}$, and putting $x = x_1$, the result is a relation similar to (79) and (80), namely,

$$\frac{\beta_1}{\beta} = (n+2)^{\frac{1}{n+1}} \qquad (86)$$

The average friction coefficient on a cone is defined as

$$C_F = \frac{D}{\frac{\rho_e u_e^2}{2} A_w} = \frac{\int_o^x r_w \tau_w dx}{\frac{\rho_e u_e^2}{2} \int_o^x r_w dx} \qquad (87)$$

or, with $r_w = ax$

$$C_F = \frac{2}{Re_x^2} \int_o^{Re_x} C_f \; Re_x \; d \; Re_x \qquad (88)$$

For flow over a flat plate

$$C_{F_1} = \frac{D}{\frac{\rho_e u_e^2}{2} x_1} = \frac{\int_o^{x_1} \tau_{w_1} dx_1}{\frac{\rho_{e_1} u_{e_1}^2}{2} x_1}$$

or

$$C_{F_1} = \frac{1}{Re_{x_1}} \int_o^{Re_{x_1}} C_{f_1} \; d \; Re_{x_1} \qquad (89)$$

From (71)

$$C_f = \frac{C_2}{Re_x^{\frac{n}{n+1}}} ,$$

then (88) gives

$$C_F = 2C_2 \; Re_x^{-\frac{n}{n+1}} \left(\frac{n+1}{n+2}\right) \qquad (90)$$

Also, from (70)

$$C_{f_1} = \frac{C_1}{Re_{x_1}^{\frac{n}{n+1}}} ,$$

then (89) gives

$$C_{F_1} = C_1 \; Re_{x_1}^{-\frac{n}{n+1}} \; (n+1) \tag{91}$$

From (90) and (91) it then follows that

$$\frac{C_F}{C_{F_1}} = 2 \; \frac{C_2}{C_1} \; \frac{Re_x^{-\frac{n}{n+1}}}{Re_{x_1}^{-\frac{n}{n+1}}} \; \frac{1}{(n+2)} \tag{92}$$

The ratio $\frac{C_2}{C_1}$ is evaluated by putting (70) and (71) into (67) with the result

$$\frac{C_2}{C_1} = (n+2)^{\frac{n}{n+1}} \tag{93}$$

Then, for $Re_{x_1} = Re_x$, (92) becomes

$$\frac{C_F}{C_{F_1}} = \frac{2}{(n+2)^{\frac{1}{n+1}}} \tag{94}$$

Equation (94) holds for equal Reynolds numbers based on distance along the surface, and for equal Mach numbers outside the boundary layer and equal $\frac{T_w}{T_e}$.

The average Stanton number on a cone is defined as

$$ST = \frac{\int_o^x r_w Q_w dx}{\rho_e u_e (I_a - I_w) \int_o^x r_w dx} \tag{95}$$

for I_a, I_w independent of x. With $r_w = ax$, (95) becomes

$$ST = \frac{2}{Re_x^{\,2}} \int_o^{Re_x} St\ Re_x\ d\ Re_x \tag{96}$$

For flow over a flat plate ST is defined by

$$ST_1 = \frac{\int_o^{x_1} Q_{w_1} dx_1}{\rho_{e_1} u_{e_1} (I_{a_1} - I_{w_1}) x_1}$$

or

$$ST_1 = \frac{1}{Re_{x_1}} \int_o^{Re_{x_1}} St_1\ d\ Re_{x_1} \tag{97}$$

for I_{a_1}, I_{w_1} independent of x_1.

For a plate or cone with constant T_w the conclusion from (48), (50), (51), (52), and (53) that the local heat transfer Q varies as $Re_L^{-\frac{n}{n+1}}$ can be expressed as

$$\frac{Q_w}{\rho_\infty u_\infty I_\infty} Re_x^{\frac{n}{n+1}} = C_3$$

or, with

$$St = \frac{Q_w}{\rho_e u_e (I_a - I_w)}$$

as

$$St\ Re_x^{\frac{n}{n+1}} = C_3\ \left(\frac{\rho_\infty}{\rho_e}\right)\left(\frac{u_\infty}{u_e}\right)\ \frac{1}{I_{*a}-I_{*w}} \tag{98}$$

Also,

$$St_1\ Re_{x_1}^{\frac{n}{n+1}} = C_4\ \left(\frac{\rho_\infty}{\rho_e}\right)_1\left(\frac{u_\infty}{u_e}\right)_1\ \frac{1}{I_{*a_1}-I_{*w_1}} \tag{99}$$

Then, from (96) and (98) it follows that

$$ST = 2C_3\ \left(\frac{\rho_\infty}{\rho_e}\right)\left(\frac{u_\infty}{u_e}\right)\ \frac{1}{I_{*a}-I_{*w}}\ Re_x^{-\frac{n}{n+1}}\ \left(\frac{n+1}{n+2}\right) \tag{100}$$

and from (97) and (99) that

$$ST_1 = C_4\ \left(\frac{\rho_\infty}{\rho_e}\right)_1\left(\frac{u_\infty}{u_e}\right)_1\frac{1}{I_{*a_1}-I_{*w_1}}\ Re_{x_1}^{-\frac{n}{n+1}}(n+1) \tag{101}$$

For $Re_x = Re_{x_1}$, (100) and (101) give

$$\frac{ST}{ST_1} = \frac{2}{n+2}\ \frac{C_3}{C_4}\ \frac{I_{*a_1}-I_{*w_1}}{I_{*a}-I_{*w}}\ \frac{\left(\frac{\rho_\infty}{\rho_e}\right)\left(\frac{u_\infty}{u_e}\right)}{\left(\frac{\rho_\infty}{\rho_e}\right)_1\left(\frac{u_\infty}{u_e}\right)_1} \tag{102}$$

From (98), (99), and (81) it follows that,

$$\frac{St}{St_1} = \frac{C_3}{C_4}\ \frac{(I_{*a_1}-I_{*w_1})}{(I_{*a}-I_{*w})}\ \frac{\left(\frac{\rho_\infty}{\rho_e}\right)\left(\frac{u_\infty}{u_e}\right)}{\left(\frac{\rho_\infty}{\rho_e}\right)_1\left(\frac{u_\infty}{u_e}\right)_1} = (n+2)^{\frac{n}{n+1}} \tag{103}$$

Then from (102) and (103)

$$\frac{ST}{ST_1} = \frac{2}{(n+2)^{\frac{1}{n+1}}} \tag{104}$$

Therefore, from (94) and (104)

$$\frac{ST}{ST_1} = \frac{C_F}{C_{F_1}} \tag{105}$$

or

$$\frac{ST}{C_F} = \frac{ST_1}{C_{F_1}} \tag{106}$$

for equal Reynolds numbers based on distance along the surface, equal M_e, and equal $\frac{T_w}{T_e}$ on cone and flat plate.

To obtain the relation between the recovery factor r on a cone and on a flat plate let the conditions at infinity for the cone boundary layer be those in an inviscid flow behind the nose shock and on the cone surface. The boundary conditions (52) and (53) then become

$$u_{*e} = 1,\ I_{*e} = 1,\ p_{*e} = \text{const},\ u_{*w} = 0,\ v_{*w} = 0.$$

Also, for the zero heat-transfer case, $Q_{*w} = 0$.

Assume that (48), (50), and (51) are then solved and $I_{*w} = I_{*a}$ obtained. Because I_{*a} is independent of Re_L it follows that for a cone I_{*a} is independent of Re_x. The transformation (12) then gives

$$I_{w_1} = I_w$$

or

$$I_{a_1} = I_a$$

The transformations (12) and (18) also give

$$I_{e_1} = I_e \quad ,$$

and

$$u_{e_1} = u_e \quad ,$$

where, for cone and plate the conditions at the boundary-layer edge are the same as those at infinity; that is $(\ )_e = (\ )_\infty$. Then $I_{*a_1} = I_{*a}$, $I_{*e_1} = I_{*e} = 1$, $u_{*e_1} = u_{*e} = 1$, and $\left(\frac{u_\infty^2}{I_\infty}\right)_1 = \frac{u_\infty^2}{I_\infty}$. Then it follows from (55) that $r_1 = r$. That is, the recovery factors for a cone and plate are equal. Moreover, because I_{*a} and I_{*a_1} are independent of Re_x, r and r_1 are independent of Re_x.

DISCUSSION

The transformation between axisymmetric and two-dimensional turbulent boundary-layer flow requires that the function $f(\frac{r_w}{L})$ in equation (16) be known. Equation (40) is a relation for $f(\frac{r_w}{L})$ but because C_{f_1} is unknown until $f(\frac{r_w}{L})$ is known the relation is not convenient. By approximating the friction coefficient by a power formula (eqs. 42 and 43) a convenient relation, (eq. 44), is obtained for $f(\frac{r_w}{L})$. For incompressible flow, power formulas are known to be good approximations for C_f. Figure II 4 of Reference 5 shows how the Young, Blasius, and Falkner power formulas closely approximate the more involved and more accurate Schoenherr, Coles, Schultz-Grunow, and Squire and Young, friction formulas over the range of Re_θ between 10^2 and 10^7.

When the relation for $f(\frac{r_w}{L})$ given by (44) is applied to a flow in which M_e, $\frac{T_w}{T_e}$, and H_u vary along the body, the exponent n remains fixed. That is, C_f must vary with Re_θ in the same way for the range of M_e, $\frac{T_w}{T_e}$, and velocity-profile shapes encountered in a particular case.

One method for obtaining C_f for compressible flow from C_f for incompressible flow is the "reference-enthalpy method" (Ref. 6). In this method, k in (42) is replaced by the product of k and a function of Mach number and wall-temperature ratio. The exponent n remains unchanged. Thus, if k and n are known for incompressible flow the relation (42) becomes

$$\frac{C_f}{2} = \frac{k}{Re_\theta^{\,n}} \left(\frac{\mu_r}{\mu_e}\right)^n \left(\frac{\rho_r}{\rho_e}\right)$$

for compressible flow (Ref. 7). Because the Mach number and wall-temperature ratio are the same at corresponding x and x_1, $(\frac{\mu_r}{\mu_e})$ and $(\frac{\rho_r}{\rho_e})$ are also the same. Moreover, introducing a function of H_u to allow for velocity-profile shape, as in the Ludwig-Tillman formula (Ref. 8),

$$\frac{C_f}{2} = \frac{.123}{Re_\theta^{\,.268}} \, 10^{-.678 H_u} ,$$

still results in (44) because H_u is the same at corresponding x and x_1 because the velocity profiles are similar at corresponding x and x_1.

In conclusion it is noted that because a power form for a friction formula is approximate, so are all the results, including those that follow from non-dimensionalization to the form (48), (50), and (51). In contrast, for laminar flow n = 1 exactly, and all the conclusions for laminar flow are exact.

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13 ABSTRACT

With the restriction that the friction coefficient is expressible as a power function of the local boundary-layer Reynolds number, the Mangler transformation between axisymmetric and two-dimensional laminar boundary-layer flow is generalized to turbulent flow. Relations are obtained between turbulent boundary-layer quantities in an axisymmetric flow and those at corresponding points in a substitute two-dimensional flow. The transformation is applied to the supersonic turbulent boundary layer on a cone with an attached shock wave and yields simple relations between boundary-layer quantities for a cone and those for a corresponding flat-plate flow. A non-dimensionalization of the equations of continuity, motion, and total-enthalpy gives the variation with Reynolds number of a number of turbulent boundary-layer quantities.

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